



Stability and Super Stability of Fuzzy Approximately Ring Homomorphisms and Fuzzy Approximately Ring Derivations

N. Eghbali

Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran

Received: 05/08/2014

Accepted: 15/02/2020

Published: 20/05/2020

Abstract

In this paper, we establish the Hyers-Ulam-Rassias stability of ring homomorphisms and ring derivations in the uniform case on fuzzy Banach algebras.

Keywords: Fuzzy normed space; Approximately ring homomorphism; Stability

1 Introduction

It seems that the stability problem of functional equations had been first raised by Ulam [12]. An answer to this problem has been given at first by Hyers [5] and then by Th. M. Rassias as follows [10]. Suppose E_1 and E_2 are two real Banach spaces and $f: E_1 \rightarrow E_2$ is a mapping. If there exist $\delta \geq 0$ and $0 \leq p < 1$ such that $\|f(x+y) - f(x) - f(y)\| \leq \delta(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$, then there is a unique additive mapping $T: E_1 \rightarrow E_2$ such that $\|f(x) - T(x)\| \leq 2\delta\|x\|^p/(2 - 2^p)$ for every $x \in E_1$. In 1991, Gajda [2] gave a solution to this question for $p > 1$. For the case $p = 1$, Th. M. Rassias and Šemrl [11] showed that there exists a continuous real-valued function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that f can not be approximated with an additive map. In 1992, Gavruta [3] generalized the result of Rassias for the admissible control functions. Moreover the approximately mappings have been studied extensively in several papers. (See for instance [6], [7]). Fuzzy notion introduced firstly by Zadeh [13] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0,1]$. Later, in 1984 Katsaras [8] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Defining the class of approximately solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated by an exact solution of the considered equation in the fuzzy Banach algebra.

To answer this question, we use here the definition of fuzzy normed spaces given in [8] to exhibit some reasonable notions of fuzzy approximately ring homomorphism in fuzzy normed algebras and we will prove that under some suitable conditions an approximately ring homomorphism f from an algebra X into a fuzzy Banach algebra Y can be approximated in a fuzzy sense by a ring homomorphism T from X to Y . Let A be a real or complex Banach algebra. A mapping $D: A \rightarrow A$ is said to be a ring derivative if

$$\begin{aligned} D(a+b) &= D(a) + D(b), & \text{for every } a, b \in A; \\ D(ab) &= D(a)b + D(b)a, & \text{for every } a, b \in A. \end{aligned}$$

It is of interest to consider an approximately ring derivation on a Banach algebra. First of all, does there exist an approximately ring derivation f which is not an exact ring derivation? If such a mapping f do exist, then it seems natural to consider the following stability problem: does there exist ring derivation near to f ? The purpose of this paper is to prove the stability of fuzzy approximately ring derivations. In fact, under a mild assumption that A is without order, we show the Bourgin-type [1] super stability result.

2 Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions. Definition 2.1 Let X be a real linear space. A function $N: X \times \mathbb{R} \rightarrow [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,

- (N1) $N(x, c) = 0$ for $c \leq 0$;
 - (N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
 - (N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
 - (N4) $N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\}$;
 - (N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t \rightarrow \infty} N(x, t) = 1$;
 - (N6) for $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.
- The pair (X, N) is called a fuzzy normed linear space.

Example 2.2 Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} 0, & t \leq 0; \\ \frac{t}{||x||}, & 0 < t \leq ||x||; \\ 1, & t > ||x||. \end{cases}$$

is a fuzzy norm on X . Definition 2.3 Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n \rightarrow \infty} x_n = x$. Definition 2.4 A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space. Let X be an algebra and (X, N) be complete fuzzy normed space. The pair (X, N) is said to be a fuzzy Banach algebra if for every $x, y \in X$ and $s, t \in \mathbb{R}$ we have $N(xy, st) \geq \min\{N(x, s), N(y, t)\}$. Example 2.5 Let $(X, ||\cdot||)$ be a Banach algebra. Define,

$$N(x, a) = \begin{cases} 0, & a \leq ||x||; \\ 1, & a > ||x||. \end{cases}$$

Then (X, N) is a fuzzy Banach algebra. Theorem 2.6 Let X be a linear space and (Y, N) be a fuzzy Banach space. Let $\varphi: X \times X \rightarrow [0, \infty)$ be a control function such that

$$\tilde{\varphi}(x, y) = \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty,$$

for all $x, y \in X$. Let $f: X \rightarrow Y$ be a uniformly approximately additive function with respect to φ in the sense that

$$\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t\varphi(x, y)) = 1$$

uniformly on $X \times X$. Then $T(x) = N - \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ for all $x \in X$ exists and defines an additive mapping $T: X \rightarrow Y$ such that if for some $\delta > 0$, $\alpha > 0$

$$N(f(x+y) - f(x) - f(y), \delta\varphi(x, y)) > \alpha,$$

for all $x, y \in X$; then

$$N(T(x) - f(x), \delta/2\tilde{\varphi}(x, x)) > \alpha,$$

for every $x \in X$. Proof. [9]. Corollary 2.7 Let X be a normed linear space and (Y, N) a fuzzy Banach space. Let $\theta \geq 0$ and $0 \leq q < 1$. Suppose that $f: X \rightarrow Y$ is a function such that

$$\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t\theta(||x||^q + ||y||^q)) = 1$$

uniformly on $X \times X$. Then there is a unique additive mapping $T: X \rightarrow Y$ such that

$$\lim_{t \rightarrow \infty} N(T(x) - f(x), \frac{2\theta t ||x||^q}{1-2^{q-1}}) = 1$$

uniformly on X . Proof. [9].

Remark 2.8 Using the sequence $\{2^n f(2^{-n}x)\}$, one can get dual version of Theorem 2.6 and Corollary 2.7 when the control function satisfies

$$\sum_{n=0}^{\infty} 2^n \varphi(2^{-n}x, 2^{-n}y) < \infty.$$

In particular, the similar results hold for $\varphi(x, y) = ||x||^q + ||y||^q$, where $q > 1$. Stability and super stability of fuzzy approximately ring homomorphisms and fuzzy approximately ring derivations (N. Eghbali Tue Dec 24 00:16:25 2019).

3 Stability of fuzzy approximately ring homomorphism

We start our work with definition of fuzzy approximately ring homomorphism. Definition 3.1 Let X be a linear algebra, (Y, N) a fuzzy Banach algebra and $\theta \geq 0$. We say that $f: X \rightarrow Y$ is a fuzzy approximately ring homomorphism map if

$$\lim_{t \rightarrow \infty} N(f(xy) - f(x)f(y), t\theta ||x||^q ||y||^q) = 1,$$

uniformly on $X \times X$. Theorem 3.2 Let X be a normed linear algebra and (Y, N) a fuzzy Banach algebra. Let $\theta \geq 0$ and $q \geq 0, q \neq 1$. Suppose that $f: X \rightarrow Y$ is a function such that

$$\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t\theta(||x||^q + ||y||^q)) = 1,$$

uniformly on $X \times X$ and

$$\lim_{t \rightarrow \infty} N(f(xy) - f(x)f(y), t\theta ||x||^q ||y||^q) = 1, \quad (3.1)$$

uniformly on $X \times X$. Then there is a unique ring homomorphism $T: X \rightarrow Y$ such that

$$\lim_{t \rightarrow \infty} N(T(x) - f(x), \frac{2\theta t ||x||^q}{|1-2^{q-1}|}) = 1$$

uniformly on X . Proof. Theorem 2.6 and Corollary 2.7 show that there exists a unique additive mapping T such that

$$\lim_{t \rightarrow \infty} N(T(x) - f(x), \frac{2\theta t ||x||^q}{|1-2^{q-1}|}) = 1, \quad (3.2)$$

where $x \in X$. Now we only need to show that T is a multiplicative function. Put $s = \frac{1-q}{|1-q|}$ and fix $x, y \in X$ arbitrarily. By (3.2) we have

$$\lim_{t \rightarrow \infty} N(n^{-s}T(n^s x) - n^{-s}f(n^s x), \frac{2\theta t n^{-s} ||n^s x||^q}{|1-2^{q-1}|}) = 1.$$

Using the additivity of T we have:

$$\lim_{t \rightarrow \infty} N(T(x) - n^{-s}f(n^s x), \frac{2\theta t n^{s(q-1)} ||x||^q}{|1-2^{q-1}|}) = 1.$$

Since $s(q-1) < 0$ we obtain $\lim_{n \rightarrow \infty} \frac{2\theta t n^{s(q-1)} ||x||^q}{|1-2^{q-1}|} = 0$. So there is some $n_1 > 0$ such that $\frac{2\theta t n^{s(q-1)} ||x||^q}{|1-2^{q-1}|} < t$ for all $n \geq n_1$ and $t > 0$. Hence $\lim_{t \rightarrow \infty} N(T(x) - n^{-s}f(n^s x), t) \geq \lim_{t \rightarrow \infty} N(T(x) - n^{-s}f(n^s x), \frac{2\theta t n^{s(q-1)} ||x||^q}{|1-2^{q-1}|})$. So

$$N - \lim_{n \rightarrow \infty} n^{-s}f(n^s x) = T(x) \quad (3.3)$$

Using (3.1) we get

$$\lim_{t \rightarrow \infty} N(f((n^s x)y) - f(n^s x)f(y), t\theta ||n^s x||^q ||y||^q) = 1,$$

for all $x, y \in X$. Thus

$$\lim_{t \rightarrow \infty} N(n^{-s} f((n^s x)y) - n^{-s} f(n^s x)f(y), t\theta n^{s(q-1)} ||x||^q ||y||^q) = 1,$$

for all $x, y \in X$. Since $\lim_{n \rightarrow \infty} t\theta n^{s(q-1)} ||x||^q ||y||^q = 0$, there is some $n_2 > 0$ such that

$$t\theta n^{s(q-1)} ||x||^q ||y||^q < t,$$

for all $n \geq n_2$. We get

$$\lim_{t \rightarrow \infty} N(n^{-s} f((n^s x)y) - n^{-s} f(n^s x)f(y), t) \geq \lim_{t \rightarrow \infty} N(n^{-s} f((n^s x)y) - n^{-s} f(n^s x)f(y), t\theta n^{s(q-1)} ||x||^q ||y||^q).$$

So

$$N - \lim_{n \rightarrow \infty} n^{-s} f((n^s x)y) = N - \lim_{n \rightarrow \infty} n^{-s} f(n^s x)f(y) \quad (3.4)$$

By putting $n_0 = \min\{n_1, n_2\}$ and applying (3.3) and (3.4) we have

$$T(xy) = N - \lim_{n \rightarrow \infty} n^{-s} f(n^s xy) = N - \lim_{n \rightarrow \infty} n^{-s} f(n^s x)f(y) = T(x)f(y),$$

for all $x, y \in X$. From this equation by the additivity of T we have:

$$T(x)f(n^s y) = T(x(n^s y)) = T(n^s x)f(y) = n^s T(x)f(y),$$

for all $x, y \in X$. Therefore $T(x)n^{-s} f(n^s y) = T(x)f(y)$. By letting n tend to infinity we see that $T(x)T(y) = T(x)f(y)$ for all $x, y \in X$. To prove the uniqueness property of T , assume that T^* is another ring homomorphism satisfying

$$\lim_{t \rightarrow \infty} N(T^*(x) - f(x), \frac{2\theta t ||x||^q}{1 - 2^{q-1}}) = 1$$

We have

$$N(T(x) - T^*(x), t) \geq \min\{N(T(x) - n^{-s} f(n^s x), t/2), N(T^*(x) - n^{-s} f(n^s x), t/2)\}$$

Given, $\varepsilon > 0$ by (3.3) we can find some $t_0 > 0$ such that

$$N(T(x) - n^{-s} f(n^s x), t/2) \geq 1 - \varepsilon,$$

and

$$N(T^*(x) - n^{-s} f(n^s x), t/2) \geq 1 - \varepsilon,$$

for all $t/2 \geq t_0$, $x \in X$ and $n \in \mathbb{N}$. So

$$N(T(x) - T^*(x), t) \geq 1 - \varepsilon,$$

for all $t > 0$. Hence by items (N5) and (N2) of definition 2.1 we have $T(x) = T^*(x)$ for all $x \in X$. In the following example we will show that Theorem 3.2 does not necessarily hold for $q = 1$.

Example 3.3 Let X be a Banach algebra, $x_0 \in X$ and α, β are real numbers such that $|\alpha| \geq 1 - ||x|| ||y||$ and $|\beta| \leq ||x||$ for every $x \in X$. Put:

$$f(x) = \alpha x + \beta x_0 ||x||, (x \in X).$$

Moreover for each fuzzy norm N on X , we have

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t(||x|| + ||y||)) \\ = N(\beta x_0(||x+y|| - ||x|| - ||y||), t(||x|| + ||y||)) \\ = N(\beta x_0, \frac{t(||x|| + ||y||)}{||x+y|| - ||x|| - ||y||}) \geq N(\beta x_0, t) (x, y \in X, t \in \mathbb{R}). \end{aligned}$$

Therefore by the item (N5) of the Definition 2.1, we get $\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t(||x|| + ||y||)) = 1$, uniformly on $X \times X$. Also,

$$\begin{aligned} N(f(xy) - f(x)f(y), t||x||||y||) &= N(\alpha xy + \beta x_0 ||xy|| - (\alpha x + \beta x_0 ||x|)(\alpha y + \beta x_0 ||y|), t||x||||y||) \\ &= N(\alpha xy + \beta x_0 ||xy|| - \alpha^2 xy - \alpha \beta x x_0 ||y|| - \alpha \beta x_0 y ||x|| - \beta^2 x_0^2 ||x||||y||, t||x||||y||) \\ &\geq \min\{N((1 - \alpha)\alpha xy, \frac{t||x||||y||}{5}), N(||xy||\beta x_0, \frac{t||x||||y||}{5}), N(\beta^2 x_0^2 ||x||||y||, \frac{t||x||||y||}{5}), \\ &N(\alpha \beta x x_0 ||y||, \frac{t||x||||y||}{5}), N(\alpha \beta x_0 y ||x||, \frac{t||x||||y||}{5})\} \end{aligned}$$

where $x \in X$ and $t \in \mathbb{R}$. Taking into account the following inequalities

$$\begin{aligned} N\left((1 - \alpha)\alpha xy, \frac{t||x||||y||}{5}\right) &= N\left(\alpha xy, \frac{t||x||||y||}{5|1-\alpha|}\right) \geq \\ N\left(\alpha xy, \frac{t}{5}\right) \end{aligned} \quad (3.5)$$

$$N(||xy||\beta x_0, \frac{t||x||||y||}{5}) = N(\beta x_0, \frac{t||x||||y||}{5||xy||}) \geq N(\beta x_0, t/5), \quad (3.6)$$

$$N(\beta^2 x_0^2 ||x||||y||, \frac{t||x||||y||}{5}) = N(\beta^2 x_0^2, t/5), \quad (3.7)$$

$$N(\alpha \beta x x_0 ||y||, \frac{t||x||||y||}{5}) = N(\alpha x x_0, \frac{t||x||}{5|\beta|}) \geq N(\alpha x x_0, t/5), \quad (3.8)$$

$$N(\alpha \beta x_0 y ||x||, \frac{t||x||||y||}{5}) = N(\alpha x_0 y, \frac{t||y||}{5|\beta|}) \geq N(\alpha x_0 y, t/5), \quad (3.9)$$

it can be easily seen that $\lim_{t \rightarrow \infty} N(f(xy) - f(x)f(y), t||x||||y||) = 1$ uniformly on $X \times X$ and therefore the conditions of Theorem 3.2 are fulfilled. Now we suppose that there exists a unique ring homomorphism T satisfying the conditions of Theorem 3.2. By the equation

$$\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t(||x|| + ||y||)) = 1 \quad (3.10)$$

for given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$$N(f(x+y) - f(x) - f(y), t(||x|| + ||y||)) \geq 1 - \varepsilon,$$

for all $x, y \in X$ and all $t \geq t_0$. By using the simple induction on n , we shall show that

$$N(f(2^n x) - 2^n f(x), t n 2^n ||x||) \geq 1 - \varepsilon. \quad (3.11)$$

putting $y = x$ in (3.10), we get (3.11) for $n = 1$. Let (3.11) holds for some positive integer n . Then

$$\min\{N(f(2^{n+1}x) - 2^{n+1}f(x), t(n+1)2^{n+1}\|x\|) \geq \\ \|2^n x\|), N(2f(2^n x) - 2^{n+1}f(x), 2tn(\|2^{n-1}x\| + \\ \|2^{n-1}x\|)) \geq 1 - \varepsilon.$$

This completes the induction argument. We observe that

$$\lim_{n \rightarrow \infty} N(T(x) - f(x), nt\|x\|) \geq 1 - \varepsilon.$$

Hence

$$\lim_{n \rightarrow \infty} N(T(x) - f(x), nt\|x\|) = 1 \quad (3.12)$$

One may regard $N(x, t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t '. So (3.12) is a contradiction with the non-fuzzy sense. This means that there is no such a T .

4 Stability and super stability of fuzzy approximately ring derivation

We start our work with definition of fuzzy approximately ring derivation. Definition 4.1 Let (X, N) be a fuzzy Banach algebra and $\theta \geq 0$. We say that $f: X \rightarrow X$ is a fuzzy approximately ring derivation if

$$\lim_{t \rightarrow \infty} N(f(xy) - xf(y) - f(x)y, t\theta\|x\|^q\|y\|^q) = 1,$$

uniformly on $X \times X$. Definition 4.2 Let (X, N) be a fuzzy Banach algebra and $\theta \geq 0$. We say that $f: X \rightarrow X$ is a fuzzy approximately Jordan derivation if

$$\lim_{t \rightarrow \infty} N(f(x^2) - 2xf(x), t\theta\|x\|^{2q}) = 1,$$

uniformly on $X \times X$. Theorem 4.3 Let (X, N) be a fuzzy Banach algebra, $\theta \geq 0$ and $q \geq 0, q \neq 1$. Suppose that $f: X \rightarrow X$ is a function such that

$$\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t\theta(\|x\|^q + \|y\|^q)) = 1,$$

uniformly on $X \times X$ and

$$\lim_{t \rightarrow \infty} N(f(xy) - xf(y) - f(x)y, t\theta\|x\|^q\|y\|^q) = 1, \quad (4.1)$$

uniformly on $X \times X$. Then there is a unique ring derivation $D: X \rightarrow X$ such that

$$\lim_{t \rightarrow \infty} N(D(x) - f(x), \frac{2\theta t\|x\|^q}{|1-2^{q-1}|}) = 1,$$

uniformly on X . Proof. Theorem 2.6 and Corollary 2.7 show that there exists a unique additive mapping D such that

$$\lim_{t \rightarrow \infty} N(D(x) - f(x), \frac{2\theta t\|x\|^q}{|1-2^{q-1}|}) = 1, \quad (4.2)$$

where $x \in X$. Now we only need to show that D is a map such that $D(xy) = xD(y) - D(x)y$ for all $x, y \in X$. Put $s = \frac{1-q}{|1-q|}$ and fix $x, y \in X$ arbitrarily. By (4.2) we have

$$\lim_{t \rightarrow \infty} N(D(n^s x) - f(n^s x), \frac{2\theta t\|n^s x\|^q}{|1-2^{q-1}|}) = 1,$$

for all $x \in X$. Thus

$$\lim_{t \rightarrow \infty} N(n^{-s}D(n^s x) - n^{-s}f(n^s x), \frac{2\theta t n^{s(q-1)}\|x\|^q}{|1-2^{q-1}|}) = 1,$$

for all $x \in X$. By the additivity of D ,

$$\lim_{t \rightarrow \infty} N(D(x) - n^{-s}f(n^s x), \frac{2\theta t n^{s(q-1)}\|x\|^q}{|1-2^{q-1}|}) = 1,$$

for all $x \in X$. Since $s(q-1) < 0$ we obtain

$$\lim_{n \rightarrow \infty} \frac{2\theta t n^{s(q-1)}\|x\|^q}{|1-2^{q-1}|} = 0. \text{ So there is some } n_1 > 0 \text{ such that } \\ \frac{2\theta t n^{s(q-1)}\|x\|^q}{|1-2^{q-1}|} < t \text{ for all } n \geq n_1 \text{ and } t > 0. \text{ Hence } \\ \lim_{t \rightarrow \infty} N(T(x) - n^{-s}f(n^s x), t) \geq \lim_{t \rightarrow \infty} N(T(x) - \\ n^{-s}f(n^s x), \frac{2\theta t n^{s(q-1)}\|x\|^q}{|1-2^{q-1}|}). \text{ So}$$

$$N - \lim_{n \rightarrow \infty} n^{-s}f(n^s x) = D(x). \quad (4.3)$$

By (4.1) we have

$$\lim_{t \rightarrow \infty} N(n^{-s}f((n^s x)y) - xf(y) - \\ n^{-s}f(n^s x)y, t\theta n^{-s}\|n^s x\|^q\|y\|^q) = 1,$$

for all $x, y \in X$. Since $s(q-1) < 0$ we obtain $\lim_{n \rightarrow \infty} t\theta n^{s(q-1)}\|x\|^q\|y\|^q = 0$. So there is some $n_2 > 0$ such that $t\theta n^{s(q-1)}\|x\|^q\|y\|^q < t$, for all $n \geq n_2$ and $t > 0$. Hence:

$$\lim_{t \rightarrow \infty} N(n^{-s}f((n^s x)y) - xf(y) - n^{-s}f(n^s x)y, t) \geq \\ \lim_{t \rightarrow \infty} N(n^{-s}f((n^s x)y) - xf(y) - \\ n^{-s}f(n^s x)y, t\theta n^{s(q-1)}\|x\|^q\|y\|^q).$$

So

$$N - \lim_{n \rightarrow \infty} n^{-s}f((n^s x)y) = N - \lim_{n \rightarrow \infty} (xf(y) + \\ n^{-s}f(n^s x)y) \quad (4.4)$$

By putting $n_0 = \min\{n_1, n_2\}$ and applying (4.3) and (4.4) we get

$$D(xy) = N - \lim_{n \rightarrow \infty} n^{-s}f(n^s(xy)) = N - \\ \lim_{n \rightarrow \infty} n^{-s}f((n^s x)y) \quad (4.5)$$

$xf(y) + N - \lim_{n \rightarrow \infty} n^{-s}f(n^s x)y = xf(y) + D(x)y$, for all $x, y \in X$. Then by (4.5) and the additivity of D , we have $xf(n^s y) + n^s D(x)y = xf(n^s y) + D(x)n^s y = D(x(n^s y)) = D((n^s x)y) = n^s xf(y) + D(n^s x)y = n^s xf(y) + n^s D(x)y$.

Therefore $xf(y) = xn^{-s}f(n^s y)$ for all $x, y \in X$. By letting n tend to infinity we see that $xf(y) = xD(y)$ for all $x, y \in X$. Combining this formula with equation (4.5) we get $D(xy) = xD(y) + D(x)y$ for all $x, y \in X$. The proof of uniqueness property of D is similar to the proof of Theorem 3.2.

Corollary 4.4 Theorem 4.3 satisfies for fuzzy approximately Jordan derivation.

Proof. As same as the proof of Theorem 4.3 we have

$$D(x) = N - \lim_{n \rightarrow \infty} n^{-s}f(n^s x). \quad (4.6)$$

A quite similar argument to the proof of Theorem 4.3 shows that $D(x^2) = N - \lim_{n \rightarrow \infty} n^{-2s}f(n^{2s}x)$. By Definition (4.2) we have

$$\lim_{t \rightarrow \infty} N(n^{-2s}f(n^{2s}x^2) - 2n^{-s}xf(n^s x), t\theta n^{-2s}\|n^s x\|^{2q}) = 1,$$

for all $x \in X$.

Since $2s(q-1) < 0$ we obtain $\lim_{n \rightarrow \infty} t\theta n^{2s(q-1)}\|x\|^{2q} = 0$.

So there is some $n_0 > 0$ such that

$t\theta n^{2s(q-1)}||x||^{2q} < t$,
 for all $n \geq n_0$ and $t > 0$. Hence

$$\lim_{t \rightarrow \infty} N(n^{-2s}f(n^{2s}x^2) - 2n^{-s}xf(n^s x), t) \geq$$

$$\lim_{t \rightarrow \infty} N(n^{-2s}f(n^{2s}x^2) - 2n^sxf(n^s x), t\theta n^{2s(q-1)}||x||^{2q}),$$

 for all $x \in X$. So

$$N - \lim_{n \rightarrow \infty} n^{-2s}f(n^{2s}x) = N - \lim_{n \rightarrow \infty} 2n^sxf(n^s x). \quad (4.7)$$

Now by using (4.6) and (4.7) we get

$$D(x^2) = N - \lim_{n \rightarrow \infty} n^{-2s}f(n^{2s}x^2) = N - \lim_{n \rightarrow \infty} 2n^sxf(n^s x) = 2xD(x),$$

for every $x \in X$.

Theorem 4.5 Let (X, N) be a fuzzy Banach algebra, $\theta \geq 0$ and $q \geq 0, q \neq 1$. Suppose that $f: X \rightarrow X$ is a function such that $\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t\theta(||x||^q + ||y||^q)) = 1$, uniformly on $X \times X$ and

$$\lim_{t \rightarrow \infty} N(f(xy) - xf(y) - f(x)y, t\theta||x||^q||y||^q) = 1, \quad (4.8)$$

uniformly on $X \times X$.

Then we have

$$x\{f(ra) - rf(a)\} = 0,$$

for every $a, x \in X$ and $r \in \mathbb{Q} - \{0\}$.

Proof. Pick $a, x \in X$ and $r \in \mathbb{Q} - \{0\}$ arbitrarily. By Theorem 4.3 there exists a unique ring derivation D such that:

$$\lim_{t \rightarrow \infty} N(D(x) - f(x), \frac{2\theta t||x||^q}{|1-2^{q-1}|}) = 1, \quad (4.9)$$

where $x \in X$. Recall that D is additive, and so it is easy to see that $D(rb) = rD(b)$ for every $b \in X$. Fix $n \in \mathbb{N}$ arbitrarily and we get

$$N(D((n^{-1}x)(ra)) - rn^{-1}xf(a) - f(n^{-1}x)ra, t/2) \geq \quad (4.10)$$

$$\min\{N(D((n^{-1}x)(ra)) - rf(n^{-1}xa), t/4), N(rf(n^{-1}xa) - rn^{-1}xf(a) - rf(n^{-1}x)a, t/4)\}.$$

By (4.9) we have

$$\lim_{n \rightarrow \infty} N(D((n^{-1}x)(ra)) - rf(n^{-1}xa), \frac{2rt\theta||n^{-1}xa||^q}{|1-2^{q-1}|}) = 1.$$

Since $\lim_{n \rightarrow \infty} \frac{2rt\theta||xa||^q}{|1-2^{q-1}|} = 0$, there is some $n_1 > 0$ such that

$$\frac{2t\theta n^{-q}||xa||^q}{|1-2^{q-1}|} < t/4 \text{ for all } n \geq n_1 \text{ and } t > 0. \text{ Hence}$$

$$N(D((n^{-1}x)(ra)) - rf(n^{-1}xa), \frac{t/4}{\frac{2rt\theta n^{-q}||xa||^q}{|1-2^{q-1}|}}) \geq N(D((n^{-1}x)(ra)) - rf(n^{-1}xa), \frac{t/4}{\frac{2rt\theta n^{-q}||xa||^q}{|1-2^{q-1}|}}). \quad (4.11)$$

Also by (4.8) we get

$$\lim_{n \rightarrow \infty} N(rf(n^{-1}xa) - rn^{-1}xf(a) - rf(n^{-1}x)a, rt\theta||n^{-1}x||^q||a||^q) = 1.$$

Since $\lim_{n \rightarrow \infty} t\theta n^{-q}||x||^q||a||^q = 0$, there is some $n_2 > 0$ such that
 $t\theta n^{-q}||x||^q||a||^q < t/4$

for all $n \geq n_2$ and $t > 0$. Hence

$$N(rf(n^{-1}xa) - rn^{-1}xf(a) - rf(n^{-1}x)a, t/4) \geq \quad (4.12)$$

$$N(rf(n^{-1}xa) - rn^{-1}xf(a) - rf(n^{-1}x)a, t\theta n^{-q}||x||^q||a||^q).$$

By using (4.9) and (4.8) for given $\varepsilon > 0$ we can find some $t_1, t_2 > 0$ such that

$$N(D((n^{-1}x)(ra)) - rf(n^{-1}xa), \frac{2rt\theta n^{-q}||xa||^q}{|1-2^{q-1}|}) \geq 1 - \varepsilon, \quad (4.13)$$

and

$$N(rf(n^{-1}xa) - rn^{-1}xf(a) - rf(n^{-1}x)a, t\theta n^{-q}||x||^q||a||^q) \geq 1 - \varepsilon,$$

for all $t \geq t_0$. So by combining (4.10), (4.11), (4.12) and (4.13) we get

$$N(D((n^{-1}x)(ra)) - rn^{-1}xf(a) - f(n^{-1}x)ra, t/2) \geq 1 - \varepsilon. \quad (4.14)$$

By applying (4.8) we have

$$\lim_{n \rightarrow \infty} N(n^{-1}xf(ra) + f(n^{-1}x)ra - f((n^{-1}x)(ra)), t\theta n^{-q}r^q||x||^q||a||^q) = 1.$$

Since $\lim_{n \rightarrow \infty} t\theta n^{-q}r^q||x||^q||a||^q = 0$, there exists $n_0 > 0$ such that
 $t\theta n^{-q}r^q||x||^q||a||^q < t/2$
 for every $n \geq n_0$ and $t > 0$. So

$$N(n^{-1}xf(ra) + f(n^{-1}x)ra - f((n^{-1}x)(ra)), t/2) \geq \quad (4.15)$$

$$N(n^{-1}xf(ra) + f(n^{-1}x)ra - f((n^{-1}x)(ra)), t\theta n^{-q}r^q||x||^q||a||^q),$$

for all $t \geq t_0$.

Given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$$N(n^{-1}xf(ra) + f(n^{-1}x)ra - f((n^{-1}x)(ra)), t\theta n^{-q}r^q||x||^q||a||^q) \geq 1 - \varepsilon, \quad (4.16)$$

for all $t \geq t_0$. So

$$N(n^{-1}xf(ra) + f(n^{-1}x)ra - f((n^{-1}x)(ra)), t/2) \geq 1 - \varepsilon. \quad (4.17)$$

Also we have:

$$N(f((n^{-1}x)(ra)) - rn^{-1}xf(a) - f(n^{-1}x)ra, t/2) \geq \quad (4.18)$$

$$\min\{N(f((n^{-1}x)(ra)) - D((n^{-1}x)(ra)), t/4), N(D((n^{-1}x)(ra)) - rn^{-1}xf(a) - f(n^{-1}x)ra, t/4)\}.$$

By (4.9) we have

$$\lim_{n \rightarrow \infty} N(f((n^{-1}x)(ra)) - D((n^{-1}x)(ra)), \frac{2t\theta r^q n^{-q}||xa||^q}{|1-2^{q-1}|}) =$$

1.

Since $\lim_{n \rightarrow \infty} \frac{2t\theta n^{-q} r^q \|ax\|^q}{|1-2^{q-1}|} = 0$, there is some $n_0 > 0$ such that $\frac{2t\theta n^{-q} r^q \|ax\|^q}{|1-2^{q-1}|} < t/4$, for all $n \geq n_0$ and $t > 0$. Hence

$$N(f((n^{-1}x)(ra)) - D((n^{-1}x)(ra)), t/4) \geq N(f((n^{-1}x)(ra)) - D((n^{-1}x)(ra)), \frac{2t\theta n^{-q} r^q \|ax\|^q}{|1-2^{q-1}|}).$$

By (4.9) for given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$$N\left(f((n^{-1}x)(ra)) - D((n^{-1}x)(ra)), \frac{2t\theta n^{-q} r^q \|ax\|^q}{|1-2^{q-1}|}\right) \geq 1 - \varepsilon, \quad (4.19)$$

for all $t \geq t_0$. Now by combining (4.14), (4.19) and (4.18) we get

$$N\left(f((n^{-1}x)(ra)) - rn^{-1}xf(a) - f(n^{-1}x)ra, \frac{t}{2}\right) \geq 1 - \varepsilon. \quad (4.20)$$

On the other hand, we have

$$N(n^{-1}x(f(ra) - rf(a)), t) \geq \quad (4.21)$$

$$\min\{N(n^{-1}xf(ra) + f(n^{-1}x)ra - f((n^{-1}x)(ra)), t/2), N(f((n^{-1}x)(ra)) - rn^{-1}xf(a) - f(n^{-1}x)ra, t/2)\}.$$

So by combining (4.21), (4.20) and (4.17) we have $N(n^{-1}x(f(ra) - rf(a)), t) \geq 1 - \varepsilon$. Hence $N(x(f(ra) - rf(a)), nt) \geq 1 - \varepsilon$. Now by using item (N2) of Definition 2.1 we get $x(f(ra) - rf(a)) = 0$. In the following Theorem we consider the conditions for super stability of approximately ring derivations. Theorem 4.6 Let (X, N) be a fuzzy Banach algebra without order, $\theta \geq 0$ and $q \geq 0, q \neq 1$. Suppose that $f: X \rightarrow X$ is a function such that

$$\lim_{t \rightarrow \infty} N(f(x+y) - f(x) - f(y), t\theta(\|x\|^q + \|y\|^q)) = 1,$$

uniformly on $X \times X$ and

$$\lim_{t \rightarrow \infty} N(f(xy) - xf(y) - f(x)y, t\theta\|x\|^q\|y\|^q) = 1, \quad (4.22)$$

uniformly on $X \times X$. Then f is a ring derivation. *Proof.* Let D be a unique ring derivation as in Theorem 4.3. Fix $n \in \mathbb{N}$ arbitrarily, and put $s = \frac{1-q}{1-2^q}$. It follows from Theorem 4.5 that $f(n^{-s}a) = n^{-s}f(a)$ is true for every $a \in X$. Since if $f(n^{-s}a_0) \neq n^{-s}f(a_0)$ for some $a_0 \in X$, then it would follow from Theorem 4.5 that

$$x\{f(n^{-s}a_0) - n^{-s}f(a_0)\} = 0$$

for all $x \in X$, which would be contradiction because X is assumed to be without order. We get from

$$\lim_{t \rightarrow \infty} N(D(x) - f(x), \frac{2\theta t\|x\|^q}{|1-2^{q-1}|}) = 1,$$

that

$$\lim_{t \rightarrow \infty} N(n^s D(n^{-s}x) - n^s f(n^{-s}x), \frac{2\theta t n^s \|n^{-s}x\|^q}{|1-2^{q-1}|}) = 1, \quad (4.23)$$

for every $x \in X$. Since $\lim_{n \rightarrow \infty} \frac{2\theta t n^s \|n^{-s}x\|^q}{|1-2^{q-1}|} = 0$, there is some $n_0 > 0$ such that $\frac{2\theta t n^s \|n^{-s}x\|^q}{|1-2^{q-1}|} < t$ for every $n \geq n_0$. So

$$N(n^s D(n^{-s}x) - n^s f(n^{-s}x), t) \geq N(n^s D(n^{-s}x) - n^s f(n^{-s}x), \frac{2\theta t n^s \|n^{-s}x\|^q}{|1-2^{q-1}|}).$$

By using (4.23) for given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$$N(n^s D(n^{-s}x) - n^s f(n^{-s}x), \frac{2\theta t n^s \|n^{-s}x\|^q}{|1-2^{q-1}|}) \geq 1 - \varepsilon,$$

for all $t \geq t_0$. Hence

$$N(D(x) - f(x), t) = 1,$$

for all $t > 0$. So by using item (N2) of Definition 2.1, we have $D(x) = f(x)$ for every $x \in X$, which shows that f is a ring derivation.

Ethical issue

Authors are aware of, and comply with, best practice in publication ethics specifically with regard to authorship (avoidance of guest authorship), dual submission, manipulation of figures, competing interests and compliance with policies on research ethics. Authors adhere to publication requirements that submitted work is original and has not been published elsewhere in any language.

Competing interests

The authors declare that there is no conflict of interest that would prejudice the impartiality of this scientific work.

Authors' contribution

All authors of this study have a complete contribution for data collection, data analyses and manuscript writing

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